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A lower bound for the connectivity of directed Euler tour transformation graphs

Xueliang Li[1]

Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, China, and Institute of Mathematics and Physics, Xinjiang University, Urumchi, Xinjiang 830046, China

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Abstract

Let D be a directed Eulerian multigraph, v be a vertex of D. We call the common value of $\mathrm{id}(v)$ and $\mathrm{od}(v)$ the degree of v, and simply denote it by d_v. Xia introduced the concept of the T-transformation for directed Euler tours and proved that any directed Euler tour (T)-transformation graph $E_u(D)$ is connected. Zhang and Guo proved that $E_u(D)$ is edge-Hamiltonian, i.e., any edge of $E_u(D)$ is contained in a Hamilton cycle of $E_u(D)$. In this paper, we obtain a lower bound

$$\sum_{v \in Q} (d_v - 1)(d_v - 2)/2$$

for the connectivity of $E_u(D)$, where $Q = \{v \in V(D) \mid d_v \geqslant 2\}$. Examples are given to show that this lower bound is in some sense best possible.

Keywords: Connectivity; Directed Euler tour; Transformation graph

1. Introduction

Let $D = (V, A)$ be a directed Eulerian multigraph. Then for any vertex v of D, we have $\mathrm{id}(v) = \mathrm{od}(v)$. We simply denote the common value by d_v and call it the *degree of* v. Let E be a directed Euler tour of D. Then E passes through each vertex v exactly d_v times. Thus we may write E as

$$x_0' v x_1 \cdots x_1' v x_2 \cdots x_2' v \cdots v x_{d_v} \cdots x_0',$$

Correspondence address: Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, China.

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where $x'_0, x'_1, \dots, x'_{d_v-1}$ are the arcs going into v and $x_1, \dots, x_{d_v}$ are the arcs going out of v. A triple (x'_{i-1}, v, x_i) is called a *transition* of E through v. A subsequence of E starting from v and ending at u (or v) which contains at least one arc is called a $v-u$ *(or $v-v$) segment* of E. Let S and S' be two arc-disjoint $v-u$ segments of E such that (S, S') is not a partition of E. We call S and S' to be *exchangeable*. A directed Euler tour F is said to be obtained from E by a *T-transformation* at S and S', denoted by $F = T(E)$, if F is obtained from E by exchanging S and S'. The *directed Euler tour graph* of D, denoted by $E_u(D)$, is an *undirected simple graph* defined as follows: The *vertices* of $E_u(D)$ are directed Euler tours of D, and two directed Euler tours E and F are *adjacent* in $E_u(D)$ if they can be obtained from each other by a T-transformation.

For more knowledge on Eulerian graphs, we refer the reader to Fleischner [1].

Xia [7] introduced the concept of the T-transformation for directed Euler tours and proved that any directed Euler tour (T-)transformation graph $E_u(D)$ is connected. Zhang and Guo [9] proved that any edge of $E_u(D)$ is contained in a Hamilton cycle of $E_u(D)$. Now we will give a lower bound for the connectivity of $E_u(D)$ and examples to show that this lower bound is in some sense best possible. First, we need the following preparations.

2. Preliminaries

Let $Q = Q(D)$ be the set of vertices v of D such that $d_v \geqslant 2$ and v is not a cut-vertex with degree 2. We assume that $Q \neq \emptyset$, for otherwise, we get a trivial case that D has only one directed Euler tour. For $v \in Q$, we denote by S_i the set of all directed Euler tours of D which contain the transition (x'_0, v, x_i). Then $1 \leqslant i \leqslant d_v$ and $S_1, S_2, \dots$ form a partition of $V(E_u(D))$. Obviously, $S_j = \emptyset$ if and only if v is a cut-vertex and $\{x'_0, x_j\}$ is an arc-cut of D. Thus, if $d_v > 2$, there are at least $d_v - 1$ non-empty S_i, and if $d_v = 2$, there are exactly two non-empty S_i. Let H_i be the subgraph of $E_u(D)$ induced by S_i. Then H_i is isomorphic to $E_u(D_i)$, where D_i is a directed Eulerian graph obtained from D in the following way: Replacing v by a pair of new vertices v' and v'' and making the arc x'_0 going into v' and x_i going out of v', whereas making the other arcs incident with v being incident with v'' in the same manner as they are incident with v.

In what follows, whenever S_i $(i = 1, 2, \dots)$ are mentioned, we mean the partition of $V(E_u(D))$ through vertex $v \in Q$ and with the starting arc x'_0 incident with v.

Lemma 1. *Let S_i and S_j be non-empty with $i \neq j$. Then for each $E \in S_i$, there is at least one $F \in S_j$ such that $T(E) = F$, i.e., E is adjacent to F in $E_u(D)$.*

Proof. Since $E \in S_i$, E contains the transition (x'_0, v, x_i). Since x_j is an arc going out of v, E must contain a transition (y', v, x_j). Hence E can be written as $x'_0 v x_i \cdots y' v x_j \cdots$. We claim that there is a vertex $u(\neq v)$ of D such that u appears in both the segment $v x_i \cdots y' v$ and the segment $v x_j \cdots x'_0 v$ of E, or v appears in the segment $v x_j \cdots x'_0 v$ of E.

Otherwise, it is not difficult to see that $\{x_0', x_j\}$ would be an arc-cut and therefore $S_j = \emptyset$, a contradiction. Therefore, we have that

$$E:\quad x_0' \underbrace{v x_i \cdots u}_{S} \cdots y' \underbrace{v x_j \cdots u}_{S'} \cdots,$$

or,

$$E:\quad x_0' \underbrace{v x_i \cdots y'}_{S} \underbrace{v x_j \cdots v}_{S'} \cdots.$$

In any case, we can use a T-transformation at S and S' (as indicated in the above) to transform E into a directed Euler tour belonging to S_j. □

In order to estimate the order of $E_u(D)$, we introduce the following lemma.

Lemma 2. $|V(E_u(D))| \geqslant \prod_{v \in Q} (d_v - 1)!$.

Proof. We use induction on $\lambda(D) = \sum_{v \in Q} d_v$ to complete the proof.

Since $Q \neq \emptyset$, $\lambda(D) \geqslant 3$. If $\lambda(D) = 3$, the conclusion holds clearly.

Suppose that the conclusion is true for any directed Eulerian graph D with $\lambda(D) \leqslant m(\geqslant 3)$. If $\lambda(D) = m + 1$, there is a vertex v in Q. Since S_i $(i = 1, 2, \ldots)$ form a partition for $V(E_u(D))$, we have that

$$|V(E_u(D))| = \sum_i |S_i| \geqslant (d_v - 1) \min\{|S_i|\} \quad \text{(at least } d_v - 1 \text{ non-empty } S_i).$$

For each S_i, since $\lambda(D_i) \leqslant m$, from the induction hypothesis we know that

$$|S_i| = |V(E_u(D_i))| \geqslant (d_v - 2)! \prod_{u \in Q(D) \setminus v} (d_u - 1)!.$$

Hence, we have

$$|V(E_u(D))| \geqslant \prod_{v \in Q} (d_v - 1)!. \qquad \square$$

From Lemma 2, we see that, generally speaking, the order of a directed Euler tour transformation graph is considerably large. Thus, it is very difficult to give such a non-trivial concrete example.

Lemma 3. *Let S_i and S_j be non-empty with $i \neq j$. Then there are at least*

$$\sum_{u \in Q \setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1$$

independent edges (edges without any common end vertices) between S_i and S_j.

Proof. We use induction on $\lambda(D)=\sum_{v\in Q} d_v$ to complete the proof.

If $\lambda(D)=3$, the conclusion is obviously true.

Suppose that the conclusion is true for any directed Eulerian graph D with $\lambda(D)\leqslant m(\geqslant 3)$. If $\lambda(D)=m+1$, there is a vertex v in Q. Consider the partition $S_1,S_2,\ldots,S_{d_v-1}$ (or S_{d_v}) of $V(E_u(D))$. We claim that there is a vertex u such that $u\in Q(D_i)$ for any $i=1,2,\ldots$. In fact, if $|Q|=1$, then $d_v=\lambda(D)=m+1>3$ and therefore $d_v\geqslant 3$ in any D_i. Thus $u=v$ is such a vertex. If $|Q|\geqslant 2$, there is a vertex $u\neq v$ such that $u\in Q$ and therefore $u\in Q(D_i)$ for any i. So, for each $i=1,2,\ldots,d_v-1$ (or d_v) we can partition $V(E_u(D_i))$ through u as $S_{i1},S_{i2},\ldots,S_{it}$ with $t=d_u-1$ (or d_u). We consider the following two cases.

Case 1: $|Q|\geqslant 2$. As stated in the above, S_i and S_j can be partitioned into $S_{i1},S_{i2},\ldots,S_{ik},\ldots,S_{i(d_u-1)}$ (or S_{id_u}) and $S_{j1},S_{j2},\ldots,S_{jk},\ldots,S_{j(d_u-1)}$ (or S_{jd_u}), respectively. For a fixed k, consider how many independent edges there are between S_{ik} and S_{jk}.

It is easy to see that we have a directed Eulerian graph D_{ik} for S_{ik} and $S_{ik}=V(E_u(D_{ik}))$. Since $S_{1k},S_{2k},\ldots,S_{ik},\ldots,S_{jk},\ldots,S_{(d_v-1)k}$ (or S_{d_vk}) form a partition of $V(D_k')$, where D_k' is a directed Eulerian graph obtained through the vertex u in a similar way to D_i, mentioned at the beginning of this section. From $\lambda(D_k')\leqslant m$ and the induction hypothesis, we know that there are at least

$$\sum_{w\in Q(D_k')\setminus v}(d_w-1)(d_w-2)/2+(d_v-2)(d_v-3)/2+1$$

independent edges between S_{ik} and S_{jk}. Now fixing i and j, running k over $1,2,\ldots,d_u-1$ (or d_u) and noticing that $S_i=\bigcup_k S_{ik}$, we know that there are at least

$$\begin{aligned}
&(d_u-1)\left\{\sum_{w\in Q(D_k')\setminus v}(d_w-1)(d_w-2)/2+(d_v-2)(d_v-3)/2+1\right\}\\
&\geqslant \sum_{w\in Q(D_k')\setminus\{u,v\}}(d_w-1)(d_w-2)/2+(d_u-2)(d_u-3)/2\\
&\quad+(d_v-2)(d_v-3)/2+(d_u-1)\\
&\geqslant \sum_{w\in Q(D)\setminus\{u,v\}}(d_w-1)(d_w-2)/2+(d_v-2)(d_v-3)/2\\
&\quad+(d_u-2)(d_u-3)/2+(d_u-2)+1\\
&\geqslant \sum_{w\in Q(D)\setminus v}(d_w-1)(d_w-2)/2+(d_v-2)(d_v-3)/2+1
\end{aligned}$$

independent edges between S_i and S_j.

Case 2: $|Q|=1$. Let $Q=\{v\}$. We need prove that there are at least $(d_v-2)(d_v-3)/2+1$ independent edges between S_i and S_j. For $d_v=4$, we can simply construct $E_u(D)$ to show the conclusion. So, we can assume that $d_v\geqslant 5$ in the following. As mentioned before, S_i and S_j can be partitioned into $S_{i1},S_{i2},\ldots,S_{i(d_v-2)}$

(or $S_{i(d_v-1)}$) and $S_{j1}, S_{j2}, \ldots, S_{j(d_v-2)}$ (or $S_{j(d_v-1)}$), respectively. For a fixed k, consider how many independent edges there are between S_{ik} and S_{jk}. In a similar discussion as in Case 1, we know that there are at least $(d_v-3)(d_v-4)/2+1$ independent edges between S_{ik} and S_{jk}. Fixing i and j, running $k = 1, 2, \ldots, d_v-2$ (or d_v-1) and noticing that $S_i = \bigcup_k S_{ik}$, we obtain that there are at least

$$(d_v-2)\{(d_v-3)(d_v-4)/2+1\} > (d_v-2)(d_v-3)/2+1$$

independent edges between S_i and S_j. □

3. Result

Theorem 1. *Let D be a directed Eulerian (multi-)graph. Then the connectivity of $E_u(D)$ is at least*

$$\sum_{v\in Q}(d_v-1)(d_v-2)/2.$$

Proof. Let $v \in Q$. Then through v we obtain a partition $S_1, S_2, \ldots, S_{d_v-1}$ (or S_{d_v}) for $V(E_u(D))$. Again we use induction on $\lambda(D) = \sum_{v\in Q} d_v$ to complete the proof.

If $\lambda(D) = 3$, the conclusion is obviously true.

Suppose that the conclusion is true for any directed Eulerian graph D with $\lambda(D) \leqslant m (\geqslant 3)$. If $\lambda(D) = m+1$, since $\lambda(D_i) \leqslant m$, by the induction hypothesis, we know that each H_i is

$$\sum_{u\in Q\setminus v}(d_u-1)(d_u-2)/2+(d_v-2)(d_v-3)/2$$

connected. By Menger's theorem, we need show that for any pair of non-adjacent vertices E and F of $E_u(D)$, there are at least $\sum_{v\in Q}(d_v-1)(d_v-2)/2$ internally disjoint paths connecting E and F in $E_u(D)$. We consider the following two cases.

Case 1: *There is an i $(1 \leqslant i \leqslant d_v)$ such that $E, F \in S_i$.* Since H_i is $\sum_{u\in Q\setminus v}(d_u-1)(d_u-2)/2+(d_v-2)(d_v-3)/2$ connected, there are such many internally disjoint paths connecting E and F in H_i. By Lemma 1, for any $j \neq i$ with $S_j \neq \emptyset$, both E and F are adjacent to some vertices in H_j. Since there are at least d_v-2 such j's and H_j is connected, there are in total at least

$$\left\{\sum_{u\in Q\setminus v}(d_u-1)(d_u-2)/2+(d_v-2)(d_v-3)/2\right\}+(d_v-2)$$
$$=\sum_{v\in Q}(d_v-1)(d_v-2)/2$$

internally disjoint paths connecting E and F in $E_u(D)$.

Case 2: $E \in S_i$ and $F \in S_j$ with $i \neq j$. We first prove that for any i and j with $i \neq j$, the subgraph of $E_u(D)$ induced by $S_i \cup S_j$ is

$$\sum_{u \in Q \setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1$$

connected. In fact, if we delete a vertex-subset C from the subgraph induced by $S_i \cup S_j$ with

$$|C| < \sum_{u \in Q \setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1$$

and the resultant subgraph is disconnected, then we shall deduce contradictions as follows.

First, there is at least one of S_i and S_j, say S_i such that $S_i \setminus C$ induces a connected subgraph of $E_u(D)$. For otherwise, since the subgraphs of $E_u(D)$ induced by S_i and S_j are H_i and H_j, respectively, which are

$$\sum_{u \in Q \setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2$$

connected, we get that

$$\begin{aligned} |C| &\geqslant 2\left\{ \sum_{u \in Q \setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 \right\} \\ &\geqslant \sum_{u \in Q \setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1, \end{aligned}$$

which contradicts the way that C was chosen.

Next, if $(S_i \cup S_j) \setminus C$ induces a disconnected subgraph of $E_u(D)$, then only the following two cases may happen.

(a) *$S_j \setminus C$ induces a disconnected subgraph of $E_u(D)$.* Let $G_1, G_2, \ldots, G_t$ be all its components. Since S_j induces a subgraph of $E_u(D)$ with connectivity at least

$$\sum_{u \in Q \setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2,$$

we have

$$|S_j \cap C| \geqslant \sum_{u \in Q \setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2.$$

Thus, $|S_i \cap C| = 0$. By Lemma 1, for each $E \in G_k$ ($k = 1, 2, \ldots, t$) there is at least one vertex in S_i adjacent to E. Hence, $(S_i \cup S_j) \setminus C$ must induce a connected subgraph of $E_u(D)$, a contradiction.

(b) *$S_j \setminus C$ induces a connected subgraph of $E_u(D)$.* On the one hand, we know that $S_i \setminus C$ must also induce a connected subgraph of $E_u(D)$; on the other hand, by Lemma 3, there are at least

$$\sum_{u \in Q \setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1$$

independent edges between S_i and S_j. However,

$$|C| < \sum_{u\in Q\setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1,$$

which implies that $(S_i \cup S_j)\setminus C$ must induce a connected subgraph of $E_u(D)$, again a contradiction.

Finally, we turn to finding $\sum_{v\in Q}(d_v - 1)(d_v - 2)/2$ internally disjoint paths connecting E and F in $E_u(D)$. Since $E \in S_i$ and $F \in S_j$ with $i \neq j$, from the above we know that $S_i \cup S_j$ induces a subgraph of $E_u(D)$ with connectivity at least

$$\sum_{u\in Q\setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1.$$

Since $E, F \in S_i \cup S_j$, there are at least

$$\sum_{u\in Q\setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1$$

internally disjoint paths connecting E and F in the subgraph induced by $S_i \cup S_j$. On the other hand, for every $k \neq i, j$ with $S_k \neq \emptyset$, by Lemma 1 we know that both E and F have some neighbors in S_k. Since H_k, the subgraph induced by S_k, is connected, there is a path connecting E and F and only passing through vertices in H_k. Since there are at least $d_v - 3$ such k, there are in total at least

$$\left\{\sum_{u\in Q\setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1\right\} + (d_v - 3)$$
$$= \sum_{v\in Q} (d_v - 1)(d_v - 2)/2$$

internally disjoint paths connecting E and F in $E_u(D)$. $\square$

Remark. In [4] or [3], we obtained the exact connectivity for the Euler tour transformation graph $E_u(G)$ of an undirected Eulerian (multi-)graph G. From [6, 2, 8] we know that $E_u(G)$ has many nice structural properties, for example, it is regular and it is the skeleton graph of a (0, 1)-polyhedron [5]. However, things are changed completely for $E_u(D)$. It is not difficult to give examples to show that $E_u(D)$ is neither regular nor the skeleton graph of any (0, 1)-polyhedron. Perhaps this is the reason why we have not yet found the exact connectivity for $E_u(D)$.

4. Concluding remark

Let $D = (V, A)$ be the directed Eulerian multigraph with $V = \{1, 2, \ldots, n\}$ and $A = \{(1,2), (2,3), \ldots, (n-1, n), (n, 1), (1, 1), (1, 1), (2, 2), (2, 2), \ldots, (n, n), (n, n)\}$. It is not difficult to see that $E_u(D) \cong K_2 \times K_2 \times \cdots \times K_2$ (n copies of K_2). Hence, the connectivity of the $E_u(D)$ is n. From the lower bound given in Theorem 1, we know that $E_u(D)$ is $\sum_{v \in V} (3 - 1)(3 - 2)/2 = n$ connected. In this sense, we say that the lower bound in Theorem 1 is best possible.

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